

Electromagnetic Induced Gravitational Perturbations

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Abstract

We study the physical consequences of two different but closely related perturbation schemes applied to the Einstein-Maxwell equations. In one case the starting space-time is flat while in the other case it is Schwarzschild. In both cases the perturbation is due to a combined electric and magnetic dipole field. We can see, within the Einstein-Maxwell equations a variety of physical consequences. They range from induced gravitational energy-momentum loss, to a well defined spin angular momentum with its loss and a center-of-mass with its equations of motion.

1 Introduction

Recently, using the spin-coefficient (SC) formalism[1], a perturbation scheme for a simple model using the Einstein-Maxwell equations was described[2, 3]. Pure electric and magnetic dipole radiation was considered as a first order perturbation off a flat space background. It was shown that this in fact leads to a second-order perturbation in the gravitational field (the Weyl tensor), and in turn a perturbed second order metric. Some interesting results were obtained by simply looking at the asymptotic Weyl tensor; these included the existence of a Bondi news function created by the dipole radiation with the accompanying classical Bondi gravitational and electromagnetic energy-loss. A pretty result was that one could identify, in the Bianchi Identities, the classical electromagnetic angular momentum loss[4].

In the present work, we apply this scheme to more complicated perturbations. We find, perturbatively, two different versions of what could be loosely characterized as generalized Reissner-Nordström space-times; that is, metrics with a mass and Coulomb charge, but now with electromagnetic dipole radiation. In one case (perturbations off Schwarzschild) we consider the mass to be zeroth order and the charge and dipole fields to be first order. In the second

case (perturbations off flat space) we consider the mass, the charge and dipole field all to be first order.

In Section II, the Maxwell equations, for both perturbation types, are first integrated. (It does not actually matter in which background [flat or Schwarzschild] this integration occurs for these two cases.) Next, (Sections III and IV) with the Maxwell field in the stress tensor as the source, we integrate the Bianchi identities obtaining the radial and non-radial behavior of the Weyl tensor. We then probe further the asymptotic behavior of the Weyl tensor, in particular looking at the angular momentum loss and Bondi energy-momentum loss theorem as well as dynamical equations for the motion of the center of mass and charge. Different physical consequences for the two different perturbations are discovered.

For completeness, in Section V the full behavior of the second order metric is presented. Sec. VI contains the discussion, while an appendix contains full expressions for the Weyl tensor and spin coefficients.

2 The Maxwell Field

We work in the Bondi coordinate system $(u, r, \zeta, \bar{\zeta})$, where u labels the light cones, \mathfrak{C} , with apex on a time-like world-line, r is the affine parameter along the null geodesics, and $\zeta = \cot(\theta/2)e^{i\phi}$ is the complex stereographic angle labeling the null geodesics on \mathfrak{C} . Furthermore, we choose the Bondi null tetrad $\{l^a, n^a, m^a, \bar{m}^a\}$ such that the vector l^a is tangent to the null geodesic congruences. This, along with the choice of l^a as a gradient, fixes the spin coefficients $\kappa, \pi, \varepsilon, \rho$, and τ as [5]: $\kappa = \pi = \varepsilon = 0$, $\rho = \bar{\rho}$, and $\tau = \bar{\alpha} + \beta$. In the Schwarzschild space-time, this fixes the set of SCs as[6]:

$$\begin{aligned} \kappa &= \pi = \varepsilon = \sigma = \tau = \nu = \lambda = 0, \\ \rho &= -r^{-1}, \\ \alpha &= -\frac{\zeta}{2r} \equiv \frac{\alpha^0}{r}, \\ \beta &= \frac{\bar{\zeta}}{2r} \equiv \frac{\beta^0}{r}, \\ \gamma &= \frac{\sqrt{2}G}{r^2 c^2} M_S \\ \mu &= -\frac{1}{r} + \frac{2\sqrt{2}G}{r^2 c^2} M_S. \end{aligned} \tag{1}$$

The flat space Minkowski set of SCs are obtained simply by setting the Schwarzschild mass, M_S , equal to zero. In both cases it can be seen that the radial and non-radial Maxwell equations are the same[5, 7], allowing us to write down the desired solution which contains both a Coulomb charge and radiating

electromagnetic dipole given by:

$$\begin{aligned}
\phi_0 &= \frac{\phi_0^0}{r^3} \equiv \frac{2D^i}{r^3} Y_{1i}^1 \\
\phi_1 &= \frac{\phi_1^0}{r^2} + \frac{\phi_1^1}{r^3} \equiv \frac{q}{r^2} + \frac{\sqrt{2}D^{i'}}{r^2} Y_{1i}^0 - \frac{D^i}{r^3} Y_{1i}^0 \\
\phi_2 &= \frac{\phi_2^0}{r} + \frac{\phi_2^1}{r^2} + \frac{\phi_2^2}{r^3} \equiv -\frac{2D^{i''}}{r} Y_{1i}^{-1} + \frac{2\sqrt{2}D^{i'}}{r^2} Y_{1i}^{-1} - \frac{D^i}{2r^3} Y_{1i}^{-1}.
\end{aligned} \tag{2}$$

The three-vector D^i is our complex dipole moment and can be written as the complex superposition of its real electric and magnetic parts:

$$D^i = D_E^i + iD_M^i, \tag{3}$$

while q is our Coulomb charge. At this point, D^i may be regarded as an arbitrary function of the retarded time, $u_r \equiv \sqrt{2}u$. In both perturbation calculations that follow, we will treat both D^i and q as being first-order quantities in the perturbation. All calculations will be performed to second order. In addition we only keep the $l = 0, 1, 2$ spherical harmonics.

Remark Other papers [8, 9, 10] have often written $D^i = q\xi^i$, where ξ^i is a complex position vector. Due to our perturbation formalism, we can make no such identification at this point in the calculation.

Remark Throughout this paper, we denote differentiation with respect to Bondi time as: $\partial_u(\cdot) = (\cdot)$, while differentiation with respect to the retarded time, $u_r = \sqrt{2}u$, is given as $\partial_{u_r}(\cdot) = (\cdot)'$. Thus, we see that $(\cdot) = \sqrt{2}(\cdot)'$. In later sections, when we want to restore units where $c \neq 1$, we must take $(\cdot)' \rightarrow c^{-1}(\cdot)$. The gravitational coupling constant is $k = 2Gc^{-4}$.

3 Schwarzschild Background Perturbation

We now consider the Maxwell field given by Eq.(2) to be a first order perturbation in the background of the Schwarzschild space-time with the spin-coefficients given by Eq.(1). In what follows, we carry this information, via the stress tensor, into the Weyl tensor by integrating the SC form of the Bianchi identities. Looking at the asymptotic Weyl tensor components we study the physical consequences (i.e., mass, momentum, angular momentum and equations of motion seen at null infinity) of our model.

3.1 The Radial and Non-radial Bianchi Identities

We are seeking a solution of the Bianchi Identities which is driven *exclusively* by the original Schwarzschild mass (zeroth order) and electromagnetic perturbation; i.e. with no gravitational degrees of freedom. This leads to $\psi_0 = 0$.

(This step has been taken by others, e.g., [2, 3]). The radial Bianchi identities are then given by [5]:

$$\frac{\partial\psi_1}{\partial r} + \frac{4\psi_1}{r} = \frac{\bar{\delta}\psi_0}{r} + \frac{5k\phi_0^0\bar{\phi}_1^0}{r^6} + \frac{k\delta(\phi_0^0\bar{\phi}_0^0) - 4k\phi_0^0\bar{\delta}\bar{\phi}_0^0}{r^7}, \quad (4)$$

$$\begin{aligned} \frac{\partial\psi_2}{\partial r} = & -\frac{3\psi_2}{r} + \frac{\bar{\delta}\psi_1}{r} - \frac{2k\phi_1^0\bar{\phi}_1^0}{r^5} + \frac{4k}{3r^6} [\phi_1^0\bar{\delta}\bar{\phi}_0^0 + \bar{\phi}_1^0\bar{\delta}\phi_0^0 - \frac{1}{2}\delta(\phi_1^0\bar{\phi}_0^0) + \frac{1}{4}\bar{\delta}(\phi_0^0\bar{\phi}_1^0)] \\ & - \frac{k}{3r^7} [\frac{5}{2}\bar{\delta}\phi_0^0\bar{\delta}\bar{\phi}_0^0 - \bar{\delta}(\bar{\phi}_0^0\bar{\delta}\phi_0^0) + \frac{1}{2}\bar{\delta}(\phi_0^0\bar{\delta}\bar{\phi}_0^0) + \phi_0^0\bar{\phi}_0^0] - k\Delta \left(\frac{\phi_0^0\bar{\phi}_0^0}{3r^6} \right) \\ & + \frac{2\sqrt{2}kG}{c^2} \frac{M_S\phi_0^0\bar{\phi}_0^0}{r^8}, \end{aligned}$$

$$\begin{aligned} \frac{\partial\psi_3}{\partial r} = & -\frac{2\psi_3}{r} + \frac{\bar{\delta}\psi_2}{r} - \frac{k\phi_2^0\bar{\phi}_1^0}{r^4} + \frac{k}{3r^5} [2\phi_2^0\bar{\delta}\bar{\phi}_0^0 + 4\bar{\phi}_1^0\bar{\delta}\phi_1^0 - \delta(\phi_2^0\bar{\phi}_0^0) + 2\bar{\delta}(\phi_1^0\bar{\phi}_1^0)] \\ & - \frac{k}{3r^6} \left[\frac{5}{2}\bar{\delta}\phi_1^0\bar{\delta}\bar{\phi}_0^0 + \frac{5}{4}\bar{\phi}_1^0\bar{\delta}^2\phi_0^0 - \bar{\delta}(\bar{\phi}_0^0\bar{\delta}\phi_1^0) + \bar{\delta}(\phi_1^0\bar{\delta}\bar{\phi}_0^0) + \bar{\delta}(\bar{\phi}_1^0\bar{\delta}\phi_0^0) \right] \\ & + \frac{k}{12r^7} [2\bar{\delta}(\bar{\delta}\phi_0^0\bar{\delta}\bar{\phi}_0^0) - \bar{\delta}(\bar{\phi}_0^0\bar{\delta}^2\phi_0^0) + 3\bar{\delta}\bar{\phi}_0^0\bar{\delta}^2\phi_0^0] - \frac{2k}{3}\Delta \left(\frac{\phi_1^0\bar{\phi}_0^0}{r^5} - \frac{\bar{\phi}_0^0\bar{\delta}\phi_0^0}{2r^6} \right) \\ & + \frac{4\sqrt{2}kG}{3c^2} M_S \left(\frac{\phi_1^0\bar{\phi}_0^0}{r^7} - \frac{\bar{\phi}_0^0\bar{\delta}\phi_0^0}{2r^8} \right), \end{aligned}$$

$$\begin{aligned} \frac{\partial\psi_4}{\partial r} = & -\frac{\psi_4}{r} + \frac{\bar{\delta}\psi_3}{r} + \frac{\bar{\delta}(\phi_2^0\bar{\phi}_1^0)}{r^4} - \frac{k}{2r^5} [\bar{\delta}(\phi_2^0\bar{\delta}\bar{\phi}_0^0) + 2\bar{\delta}(\bar{\phi}_1^0\bar{\delta}\phi_1^0) - 2\phi_2^0\bar{\phi}_0^0] \\ & + \frac{k}{4r^6} [2\bar{\delta}(\bar{\delta}\phi_1^0\bar{\delta}\bar{\phi}_0^0) + \bar{\delta}(\bar{\phi}_1^0\bar{\delta}^2\phi_0^0) - 4\bar{\phi}_0^0\bar{\delta}\phi_1^0] - \frac{k}{8r^7} [\bar{\delta}(\bar{\delta}\bar{\phi}_0^0\bar{\delta}^2\phi_0^0) - 2\bar{\phi}_0^0\bar{\delta}^2\phi_0^0] \\ & - k\Delta \left(\frac{\phi_2^0\bar{\phi}_0^0}{r^4} - \frac{\bar{\phi}_0^0\bar{\delta}\phi_1^0}{r^5} + \frac{\bar{\phi}_0^0\bar{\delta}^2\phi_0^0}{4r^6} \right) - \frac{2\sqrt{2}kG}{c^2} M_S \left(\frac{\phi_2^0\bar{\phi}_0^0}{r^6} - \frac{\bar{\phi}_0^0\bar{\delta}\phi_1^0}{r^7} + \frac{\bar{\phi}_0^0\bar{\delta}^2\phi_0^0}{4r^8} \right). \end{aligned}$$

The differential operator Δ is given by: $\Delta \equiv \partial_u - \partial_r + r^{-2}c^{-2}2\sqrt{2}GM_S\partial_r$.

Integration produces the following results. (We relegate the complicated terms, \mathcal{A}_i , involving higher r -dependence to the appendix, as these are largely unnecessary for the calculations of particular interest in this paper):

$$\psi_1 = \frac{\psi_1^0}{r^4} + \mathcal{A}_1 \quad (5)$$

$$\psi_2 = \frac{\psi_2^0}{r^3} + \mathcal{A}_2 \quad (6)$$

$$\psi_3 = \frac{\psi_3^0}{r^2} + \mathcal{A}_3 \quad (7)$$

$$\psi_4 = \frac{\psi_4^0}{r} + \mathcal{A}_4. \quad (8)$$

Here, the $\psi_1^0, \psi_2^0, \psi_3^0$ and ψ_4^0 are r - independent functions of integration, which can be determined from the non-radial Bianchi identities, which take the form [5] :

$$\delta\psi_1^0 = 3k\phi_0^0\bar{\phi}_2^0, \quad (9)$$

$$\dot{\psi}_1^0 = -\delta\psi_2^0 + 2k\phi_1^0\bar{\phi}_2^0, \quad (10)$$

$$\dot{\psi}_2^0 = -\delta\psi_3^0 + k\phi_2^0\bar{\phi}_2^0, \quad (11)$$

$$\dot{\psi}_3^0 = -\delta\psi_4^0. \quad (12)$$

The integration of Eqs.(9)-(12) is relatively straight forward, basically entailing the comparison of spherical harmonic coefficients. We make considerable use of Clebsh-Gordon expansions [11] of the quadratic terms in the Maxwell field. This process applied to Eq.(9) yields:

$$\psi_1^0 = 3kD^i\bar{D}^{j''}Y_{2ij}^1 + \psi_1^{0k}Y_{1k}^1. \quad (13)$$

The vector $\psi_1^{0k}(u_r)$ emerges as an unknown integration factor in much the same way that ψ_1^0 emerged from the integration of the radial equations. Shortly, however, we will be able to determine ψ_1^{0k} up to constants. From Eq.(10), we find:

$$\begin{aligned} \psi_2^0 = & -\frac{2\sqrt{2}G}{c^2}M_S + \Upsilon_\epsilon + \left(\frac{\sqrt{2}}{2}\psi_1^{0k'} + 2ki\bar{D}^{i''}D^{j'}\epsilon_{ijk}\right)Y_{1k}^0 \\ & + 2kq\bar{D}^{k''}Y_{1k}^0 + \sqrt{2}k\left(\frac{(D^i\bar{D}^{j''})'}{2} + \frac{\bar{D}^{i''}D^{j'}}{3}\right)Y_{2ij}^0, \end{aligned} \quad (14)$$

where the $l = 0$ function $\Upsilon_\epsilon(u_r)$ is a (first-order) function of integration. Considering Eq.(11), we recognize that as ψ_3^0 is a spin weighted $s = -1$ quantity it has no $l = 0$ harmonic contribution. This allows us to both obtain ψ_3^0 as well as place a restriction on Υ_ϵ :

$$\begin{aligned} \psi_3^0 = & \left(\sqrt{2}ki\bar{D}^{i''}D^{j''}\epsilon_{ijk} + 2\sqrt{2}ki(D^{i''}\bar{D}^{j''})'\epsilon_{ijk} - \psi_1^{0k''}\right)Y_{1k}^{-1} \\ & - 2\sqrt{2}kq\bar{D}^{k'''}Y_{1k}^{-1} + k\left(\frac{1}{3}\bar{D}^{i''}D^{j''} - (D^i\bar{D}^{j''})'' - \frac{2}{3}(D^{i''}\bar{D}^{j''})'\right)Y_{2ij}^{-1}, \end{aligned} \quad (15)$$

$$\dot{\Upsilon}_\epsilon = \sqrt{2}\Upsilon_\epsilon' = \frac{4k}{3}D^{i''}\bar{D}^{j''}\delta_{ij}. \quad (16)$$

Finally, we determine both ψ_1^{0k} and ψ_4^0 from Eq.(12). We recall that ψ_4^0 is an $s = -2$ quantity and hence does not contain an $l = 1$ harmonic. The $l = 1$ harmonic contributions to the equation thus must vanish yielding a differential condition on $\psi_1^{0k}(u_r)$; the remaining part yields the determination of ψ_4^0 :

$$\psi_4^0 = \sqrt{2}k \left[(D^i \bar{D}^{j''})''' + \frac{2}{3}(D^{i'} \bar{D}^{j''})'' - \frac{1}{3}(\bar{D}^{i''} D^{j''})' \right] Y_{2ij}^{-2}, \quad (17)$$

$$\psi_1^{0k'''} = \sqrt{2}ki[(\bar{D}^{i''} D^{j''})' \epsilon_{ijk} + 2(D^{i'} \bar{D}^{j''})'' \epsilon_{ijk} + 2iq \bar{D}^{k''''}]. \quad (18)$$

For later use we decompose Eq.(18) into its real and imaginary parts using $\psi_1^{0k} = \psi_{1R}^{0k} + i\psi_{1I}^{0k}$:

$$\psi_{1R}^{0k'''} = 2\sqrt{2}k(D_M^{j'} D_E^{i'})''' \epsilon_{ijk} - \sqrt{2}k(D_E^{i''} D_M^{j''})' \epsilon_{ijk} - 2\sqrt{2}kq D_E^{k''''} \quad (19)$$

$$\psi_{1I}^{0k'''} = 2\sqrt{2}k[(D_E^{i'} D_E^{j''} + D_M^{i'} D_M^{j''}) \epsilon_{ijk} + q D_M^{k''}]. \quad (20)$$

Note that the latter equation can be immediately integrated twice as

$$\psi_{1I}^{0k'} = 2\sqrt{2}k[(D_E^{i'} D_E^{j''} + D_M^{i'} D_M^{j''}) \epsilon_{ijk} + q D_M^{k''}], \quad (21)$$

where we have taken the two constants of integration to vanish while the first can be integrated once as:

$$\psi_{1R}^{0k''} = \sqrt{2}k[2(D_M^{j'} D_E^{i'})'' - D_E^{i''} D_M^{j''}] \epsilon_{ijk} - 2\sqrt{2}kq D_E^{k''''}. \quad (22)$$

Remark: Bramson [3], improperly set the $\psi_1^{0k} = 0$, apparently overlooking this differential condition.

Remark: There is a further equation (a reality condition) on ψ_2^0 that plays a very important role. It however involves a further variable (the spin-coefficient σ^0 , i.e., the Bondi shear) for its description.

3.2 Reality Conditions

To more easily see the physical content in our equations, we now introduce the c in all time derivatives, i.e., $(') \rightarrow c^{-1}(')$.

We turn to the Bondi mass aspect where its reality forces further restrictions on our variables. To proceed, we must first know the shear i.e., the spin coefficient σ . A more detailed discussion of the spin coefficients is given in Section V and the Appendix, but it can be seen [2] that the behavior of σ does not change with the switch from flat-space to a Schwarzschild background or the addition of a Coulomb charge. Thus, we find[2]:

$$\sigma = \frac{\sigma^0}{r^2}, \quad (23)$$

$$\psi_4^0 = -2c^{-2} \bar{\sigma}^{0''} \quad (24)$$

$$\sigma^0 = \frac{\sqrt{2}k}{2c^3} \left[\frac{1}{3} \int (D^{i''} \bar{D}^{j''}) du_r - (\bar{D}^i D^{j''})' - \frac{2}{3}(\bar{D}^{i'} D^{j''}) \right] Y_{2ij}^2. \quad (25)$$

Using Eqs.(23) and (25), the Bondi mass aspect[12, 5] is given by

$$\Psi = \psi_2^0 + \delta^2 \bar{\sigma}^0 + \sqrt{2}c^{-1} \sigma^0 (\bar{\sigma}^0)', \quad (26)$$

and is subject to the reality condition:

$$\Psi = \bar{\Psi}. \quad (27)$$

As can be seen from Eq.(25), the quantity σ^0 is second order in the perturbation, so that it can be neglected in the definition of the mass aspect in Eq.(26). If we expand the mass aspect in spherical harmonics,

$$\Psi = \Psi^0 + \Psi^i Y_{1i}^0 + \Psi^{ij} Y_{2ij}^0 + \dots,$$

then from Bondi[12, 5], the $l = 0$ and $l = 1$ terms, i.e.,

$$\Psi^0 = -\frac{2\sqrt{2}G}{c^2} M_S + \Upsilon_\epsilon = -\frac{2\sqrt{2}G}{c^2} M_B \quad (28)$$

$$\Psi^i = -\frac{6G}{c^3} P^i \quad (29)$$

are, up to numerical factors, interpreted as the Bondi mass and linear three-momentum respectively.

From Eq.(26) using Eqs.(14) and (25), we have that:

$$\begin{aligned} \Psi = & -\frac{2\sqrt{2}G}{c^2} M_S + \Upsilon_\epsilon + \left(\frac{\sqrt{2}}{2c} \psi_1^{0k'} + \frac{2ki}{c^3} \bar{D}^{i''} D^{j'} \epsilon_{ijk} \right) Y_{1k}^0 \\ & + \frac{2kq}{c^2} \bar{D}^{k''} Y_{1k}^0 + \frac{\sqrt{2}k}{6c^3} \int (D^{i''} \bar{D}^{j''}) du_r Y_{2ij}^0. \end{aligned} \quad (30)$$

By the symmetry on the $l = 2$ contribution, it follows that Ψ^{ij} is real. From the $l = 0$ contribution, we see that the Bondi mass is real:

$$-\frac{2\sqrt{2}G}{c^2} M_S + \Upsilon_\epsilon = -\frac{2\sqrt{2}G}{c^2} M_S + \bar{\Upsilon}_\epsilon = -\frac{2\sqrt{2}G}{c^2} M_B. \quad (31)$$

The reality condition on the $l = 1$ condition is a bit more complicated. First, we decompose the dipole D^i into its electric and magnetic parts,

$$D^i = D_E^i + iD_M^i,$$

and write: $\psi_1^{0k} = \psi_{1R}^{0k} + i\psi_{1I}^{0k}$. Then the condition for $\Psi^i = \bar{\Psi}^i$ yields two relations; one on the real and one on the imaginary part of ψ_1^{0k} . We find for the vanishing of the imaginary part ψ_{1I}^{0k} ,

$$\psi_{1I}^{0k'} = 2\sqrt{2}kc^{-1}qD_M^{k''} + 2\sqrt{2}kc^{-2}(D_E^{i'} D_E^{j''} + D_M^{i'} D_M^{j''})\epsilon_{ijk}, \quad (32)$$

which is identical to the earlier derived, Eq.(21). The real part leads to an expression for the Bondi linear momentum

$$\Psi^k = -\frac{6G}{c^3}P^k = \left[\frac{\sqrt{2}}{2c}\psi_{1R}^{0k} + \frac{2kq}{c^2}D_E^{k'} - \frac{2k}{c^3}(D_M^{j'}D_E^{i'})\epsilon_{ijk} \right]', \quad (33)$$

or

$$P^k = -\left[\frac{c^2\sqrt{2}}{12G}\psi_{1R}^{0k'} + \frac{2q}{3c^3}D_E^{k''} - \frac{2}{3c^4}(D_M^{j'}D_E^{i'})'\epsilon_{ijk} \right]. \quad (34)$$

By taking the u_r derivative and simplifying via Eq.(22), we obtain the electromagnetic momentum flux law,

$$P^{k'} = \frac{1}{3}c^{-4}D_E^{i''}D_M^{j''}\epsilon_{ijk}. \quad (35)$$

The imaginary part of the $l = 1$ harmonic of ψ_1^0 , i.e., ψ_{1I}^{0k} , is often, *in vacuum linear theory*, taken as proportional to the total source angular momentum as viewed from infinity. This becomes modified[10] in the presence of a Maxwell Field as

$$\psi_{1I}^{0k} - 2\sqrt{2}kqc^{-1}D_M^{k'} = -\frac{6\sqrt{2}G}{c^3}J^k. \quad (36)$$

Eq.(32) is seen as the *classical law of conservation of angular momentum for electromagnetic dipole radiation*[4] or angular momentum flux law:

$$J^{k'} = \frac{2}{3c^3}(D_E^{i''}D_E^{j'} + D_M^{i''}D_M^{j'})\epsilon_{ijk}. \quad (37)$$

3.3 Null Rotations and Equations of Motion

We now construct a transformation (a *null tetrad rotation around n^a*) to what we define as the *complex center of mass*[10].

Though this is not the place to go into a detailed explanation [13, 10, 9] of the meaning of the term '*complex center of mass*', nor into the details how it can be calculated or found, a brief explanation is in order.

In a given asymptotically flat space-time, the family of regular asymptotically shear-free null geodesic congruences are determined by 1. the Bondi asymptotic shear $\sigma^0(u_r, \zeta, \bar{\zeta})$ and 2. an arbitrary choice of a complex world-line that 'lives' in the space of complex Poincare transformations (complex Minkowski space), a subgroup of the BMS group acting on \mathcal{I}^+ . {That an asymptotically shear free null geodesic congruence picks out a complex world-line in complex Minkowski space is the central fact in the present discussion. That it has led to a series of remarkable results[10, 14] is the defense[15] of its relevance.} A particular complex world-line can be chosen so that the asymptotically defined center of mass and angular momentum both vanish on it. The basic variable used to describe the asymptotically shear-free null geodesic congruence is a (stereographic) angle field, $L(u_r, \zeta, \bar{\zeta})$ on \mathcal{I}^+ that points backwards into the space-time determining a past null direction at each point $(u_r, \zeta, \bar{\zeta})$ of \mathcal{I}^+ . The

angle field, for asymptotically shear-free congruences, satisfies the differential equation [13]

$$\delta L + LL_{,u} = \sigma^0. \quad (38)$$

The solution to Eq.(38), (accurate to our working order), is

$$L(u_r, \zeta, \bar{\zeta}) = \xi^i Y_{1i}^1 - \frac{i}{2} \xi^i v^i \epsilon_{ijk} Y_{1k}^1 + \dots, \quad (39)$$

where $\xi^i(u_r)$ and $v^i = \xi^{i'}$ are respectively the arbitrary complex world-line and its velocity. Quadratic terms and high harmonics [10] have been omitted.

The transformation (null rotation) on \mathcal{I}^+ of the Bondi tetrad (l, n, m, \bar{m}) to a new tetrad $(l^*, n^*, m^*, \bar{m}^*)$ (where the l^* is the null tangent vector to the asymptotically shear-free null geodesic congruence) is given by

$$\begin{aligned} l^* &= l + \frac{L}{r} \bar{m} + \frac{\bar{L}}{r} m + \mathcal{O}(r^{-2}) \\ m^* &= m + \mathcal{O}(r^{-1}) \\ n^* &= n, \end{aligned} \quad (40)$$

Remark We have gone from the Bondi tetrad frame with null vector l that is twist free but has shear to a null vector l^* that is asymptotically shear-free but now possesses twist. The twist, Σ , given by

$$i\Sigma = \frac{1}{2}(\delta \bar{L} + \sqrt{2} L \bar{L}' - \bar{\delta} L - \sqrt{2} \bar{L} L').$$

It vanishes, in our approximation, when the world-line ξ^i is real.

This transformation induces a transformation of the asymptotic Weyl tensor components. In particular the Weyl component ψ_1^0 transforms as

$$\psi_1^{0*} = \psi_1^0 - 3L\psi_2^0 + \dots, \quad (41)$$

The *basic physical idea* is that in linear theory the $l = 1$ harmonic component of ψ_1^0 is (usually) taken as proportional to the complex center of mass, i.e., the (real) *center of mass* + *i angular momentum*. Our procedure is now to chose the arbitrary complex world-line so that the $l = 1$ harmonic component of ψ_1^{0*} vanishes. *The world-line so obtained is the complex center of mass.* We thus have to solve

$$0 = \psi_{1i}^0 - 3L\psi_2^0|_i + \dots \quad (42)$$

for $\xi^i(u_r)$.

Using Eqs.(13), (14), and (39), Eq.(42) becomes:

$$0 = \psi_1^{0k} + \frac{6\sqrt{2}G}{c^2} M_S \xi^k - \frac{3i\sqrt{2}G}{c^2} M_S \xi^i v^j \epsilon_{ijk}. \quad (43)$$

From Eqs. (18), (32), and (33), we saw that at least up to initial conditions, the vector ψ_1^{0k} is a second order quantity in our perturbation framework. However

by choosing *time-independent first-order initial conditions*, $\psi_{1(0)}^{0k}$, we see from Eq.(43), that

$$\xi^k = -\frac{\sqrt{2}c^2\psi_{1(0)}^{0k}}{12GM_S}, \quad (44)$$

i.e., ξ^k is a constant vector and complex center of mass is at rest.

In this case, where ξ^k is a constant vector, the real part can be set to zero by a Poincaré translation on \mathfrak{I}^+ so that

$$\xi^k = \xi_I^k = -\frac{c^2\psi_{1(0)}^{0k}}{6\sqrt{2}GM_S}. \quad (45)$$

Thus, we see that when the mass of the system is considered to be zeroth order, the equation of motion for the center of mass is trivial: the center of mass simply sits on the time axis. Physically, this can be thought of in the following manner: if M_S is the initial Schwarzschild mass, then it is too “heavy” for its motion to be affected by the “small” electromagnetic perturbation given by Eq.(2) (at least to second order in the calculation). We see later that for small mass the situation is very different.

3.4 Physical Interpretations

It turns out that there are a variety of physical interpretations - some new and some old - to the results of this section.

First of all when the appropriate units are inserted, Eq.(16) is exactly the Bondi mass/energy loss equation

$$M'_B = -\frac{2}{3c^5}(D_E^{i''}D_E^{i''} + D_M^{i''}D_M^{i''}). \quad (46)$$

At this approximation it coincides with the classical electromagnetic dipole energy loss. At another approximation level there would be a forth-order correction for gravitational energy loss via the square of the Bondi news function, i.e., quadrupole radiation.

The Bondi momentum loss, given by Eq.(35),

$$P^{k'} = \frac{1}{3c^4}D_E^{i''}D_M^{j''}\epsilon_{ijk} \quad (47)$$

is just the electromagnetic momentum flux.

More interesting is Eq.(32), (or (21)):

$$\psi_{1I}^{0k'} = 2\sqrt{2}kqc^{-1}D_M^{k''} + 2\sqrt{2}kc^{-2}(D_E^{i''}D_E^{j''} + D_M^{i''}D_M^{j''})\epsilon_{ijk}. \quad (48)$$

The imaginary part of the $l = 1$ harmonic of ψ_1^0 , i.e., ψ_{1I}^{0k} , is often, *in vacuum linear theory*, taken as proportional to the total source angular momentum, J^k , as viewed from infinity. This becomes modified[10] in the presence of a Maxwell Field as

$$J^k = -\frac{c^3}{6\sqrt{2}G}\psi_{1I}^{0k} + \frac{2q}{3c^2}D_M^{k'}. \quad (49)$$

Eq.(48) is seen as the *classical law of conservation of angular momentum*[4] *for electromagnetic dipole radiation*:

$$J^{k'} = \frac{2}{3c^3}(D_E^{i''}D_E^{j'} + D_M^{i''}D_M^{j'})\epsilon_{ijk}. \quad (50)$$

In the past, the imaginary part of the world-line vector, ξ_I^i , has been identified[16, 10] with the intrinsic spin associated with the asymptotic metric via the relationship

$$S^i = M_S c \xi_I^i. \quad (51)$$

We then see, from Eq.(44), that the initial value of ψ_1^{0k} , i.e., $\psi_{1(0)}^{0k}$ is proportional to the (constant) spin

$$M_S c \xi_I^k = -\frac{c^3 \psi_{1(0)}^{0k}}{6\sqrt{2}G} = S^k. \quad (52)$$

so that the angular momentum, J^k , in Eq.(49), consists of the sum of three terms, the intrinsic spin, S^i , a complicated term involving the integral over quadratic derivatives of the dipole moments and an unusual term proportional to the derivative of the magnetic dipole moment, $\frac{2}{3}c^{-1}qD_M^{k'}$.

4 Minkowski Background Perturbation

We now consider an alternative perturbation scheme, namely perturbations off Minkowski space-time. The gravitating mass now enters into the calculation as a first order quantity in the mass term of the Weyl tensor component ψ_2^0 . In the previous model the mass was large; now it will be small. Most of the calculations of the previous section are essentially identical in this framework, so we can proceed quickly toward the null rotation calculations where new results do appear.

4.1 Radial and Non-radial Bianchi Identities

Once again, we set $\psi_0 = 0$ so that our solution is driven purely by the perturbative quantities. The radial Bianchi identities are simply those in flat Minkowski space-time. The results of their integration are the same as those obtained earlier[2] (but now with the addition of a Coulomb charge) or by setting $M_S = 0$ in Eqs.(4).

At the leading order in r , this yields radial behavior equivalent to that found in the preceding section. The non-radial integrations give virtually identical results:

$$\psi_1^0 = \psi_1^{0k} Y_{1k}^1 + 3kc^{-2} D^i \bar{D}^{j''} Y_{2ij}^1, \quad (53)$$

$$\begin{aligned}\psi_2^0 &= \Upsilon_\epsilon + \left(\frac{2kq}{c^2} \bar{D}^{k''} + \frac{\sqrt{2}}{2c} \psi_1^{0k'} + \frac{2ki}{c^3} \bar{D}^{i''} D^{j'} \epsilon_{ijk} \right) Y_{1k}^0 \\ &\quad + \frac{\sqrt{2}k}{c^3} \left(\frac{(D^i \bar{D}^{j''})'}{2} + \frac{\bar{D}^{i''} D^{j'}}{3} \right) Y_{2ij}^0,\end{aligned}\quad (54)$$

$$\begin{aligned}\psi_3^0 &= \left(\frac{\sqrt{2}ki}{c^4} \bar{D}^{i''} D^{j''} \epsilon_{ijk} + \frac{2\sqrt{2}ki}{c^4} (D^{i'} \bar{D}^{j''})' \epsilon_{ijk} - c^{-2} \psi_1^{0k''} \right) Y_{1k}^{-1} \\ &\quad - \frac{2\sqrt{2}kq}{c^2} \bar{D}^{k''} Y_{1k}^{-1} + kc^{-4} \left(\frac{1}{3} \bar{D}^{i''} D^{j''} - (D^i \bar{D}^{j''})'' - \frac{2}{3} (D^{i'} \bar{D}^{j''})' \right) Y_{2ij}^{-1},\end{aligned}\quad (55)$$

$$\dot{\Upsilon}_\epsilon = \sqrt{2} \Upsilon'_\epsilon = \frac{4k}{3c^3} D^{i''} \bar{D}^{j''} \delta_{ij}, \quad (56)$$

$$\psi_4^0 = \sqrt{2}kc^{-5} \left((D^i \bar{D}^{j''})''' + \frac{2}{3} (D^{i'} \bar{D}^{j''})'' - \frac{1}{3} (\bar{D}^{i''} D^{j''})' \right) Y_{2ij}^{-2}, \quad (57)$$

$$\psi_1^{0k''} = \sqrt{2}kc^{-2} i [(\bar{D}^{i''} D^{j''}) \epsilon_{ijk} + 2(D^{i'} \bar{D}^{j''})' \epsilon_{ijk} + 2iqc \bar{D}^{k'''}]. \quad (58)$$

We have denoted the mass term of the perturbation as Υ_ϵ to indicate that it is a first-order, *perturbative* quantity which arises as an integrating factor in ψ_2^0 . The absence of the Schwarzschild mass is the essential change from the previous section. We now obtain the reality conditions for this new calculation.

4.2 Reality Conditions

As ψ_4^0 has not changed from that obtained in Section 3.1, it follows that the value of the spin coefficient σ^0 will remain that given in Eq.(25), so the Bondi mass aspect is simply given as:

$$\begin{aligned}\Psi &= \Upsilon_\epsilon + \left(\frac{2kq}{c^2} \bar{D}^{k''} + \frac{\sqrt{2}}{2c} \psi_1^{0k'} + \frac{2ki}{c^3} \bar{D}^{i''} D^{j'} \epsilon_{ijk} \right) Y_{1k}^0 \\ &\quad + \frac{\sqrt{2}k}{6c^3} \int (D^{i''} \bar{D}^{j''}) du_r Y_{2ij}^0,\end{aligned}\quad (59)$$

and consequently, the reality conditions are little changed from those of Section 3.2. Once again, the $l = 2$ condition is trivially satisfied, while the Bondi mass and linear momentum are given respectively by:

$$\Upsilon_\epsilon = \bar{\Upsilon}_\epsilon = -\frac{2\sqrt{2}G}{c^2} M_B \quad (60)$$

$$\Psi^k = -\frac{6G}{c^3} P^k = \left[\frac{2kq}{c^2} D_E^{k'} + \frac{\sqrt{2}}{2c} \psi_{1R}^{0k} - \frac{2k}{c^3} (D_M^{j'} D_E^{i'}) \epsilon_{ijk} \right]'. \quad (61)$$

The reality of the $l = 1$ coefficient of Eq.(59) again yields:

$$\psi_{1I}^{0k'} = 2\sqrt{2}kqc^{-1}D_M^{k''} - 2\sqrt{2}kc^{-2}(D_E^{i''}D_E^{j'} + D_M^{i''}D_M^{j'})\epsilon_{ijk}. \quad (62)$$

Aside from Eq.(60) these are identical to those obtain in the prior section.

4.3 Null Rotations and Equations of Motion

It is in our attempts to understand the physical content of our equations, i.e., conservation laws, the definition of angular momentum and spin, and equations of motion for complex center of mass and charge world-lines that the present perturbation scheme departs from that of the Schwarzschild background. In particular, using the same stereographic angle field $L(u, \zeta, \bar{\zeta})$ described in Eq.(39) and applying it to the null rotation of ψ_{1i}^0 given in Eq.(42), we obtain (since M_B, q and ξ^k are first order) the *second order relation*:

$$0 = \psi_1^{0k} + \frac{6\sqrt{2}G}{c^2}M_B\xi^k \quad (63)$$

or decomposed as

$$\psi_{1R}^{0k} = -\frac{6\sqrt{2}G}{c^2}M_B\xi_R^k \quad (64)$$

$$\psi_{1I}^{0k} = -\frac{6\sqrt{2}G}{c^2}M_B\xi_I^k \quad (65)$$

Taking the prime derivative of Eq.(65) and inserting it into Eq.(62) we obtain after simplifications, again the classical conservation law of angular momentum[4],

$$\begin{aligned} (M_Bc\xi_I^k + \frac{2}{3}c^{-2}qD_M^{k'})' &= \frac{2}{3c^4}(D_E^{i''}D_E^{j'} + D_M^{i''}D_M^{j'})\epsilon_{ijk} \\ J^{k'} &= \text{ang.mom.flux} \end{aligned} \quad (66)$$

with the identifications

$$S^k = M_Bc\xi_I^k, \quad (67)$$

$$J^k = S^k + \frac{2q}{3c}D_M^{k'}. \quad (68)$$

This can be considered as the evolution equation for the spin, S^k .

In order to obtain the dynamical law for the (real) center of mass ξ_R^k , we first note, from Eq.(64), that

$$M_B\xi_R^{k''} = -\frac{c^2}{6\sqrt{2}G}\psi_{1R}^{0k''}.$$

Then, with the use of Eq.(22), i.e.,

$$\psi_1^0{}_{k''} = \sqrt{2}kc^{-2}[2(D_M^{j'}D_E^{i'})'' - D_E^{i''}D_M^{j''}]\epsilon_{ijk} - 2\sqrt{2}kc^{-1}qD_E^{k'''}, \quad (69)$$

it becomes

$$M_B\xi_R^{k''} = -\frac{2}{3c^4}[(D_E^{i'}D_M^{j'})'' - \frac{1}{2}D_E^{i''}D_M^{j''}]\epsilon_{ijk} + \frac{2q}{3c^{-3}}D_E^{k'''}, \quad (70)$$

a 2^{nd} order differential equation for ξ_R^k , an equation very much resembling Newton's 2^{nd} law.

Remark Note that if the electric dipole moment had the form $D_E^k = q\xi_R^k$, i.e., if the center of charge was the same as the center of mass, the last term would be exactly the classical radiation reaction force [14].

Returning to the Bondi linear momentum, Eq.(61), after simplification and the use of Eq.(64), becomes a dynamical expression for the momentum:

$$P^k = M_B\xi_R^{k'} - \frac{2}{3}c^{-3}qD_E^{k''} + \frac{2}{3}c^{-4}(D_M^{j'}D_E^{i'})'\epsilon_{ijk}. \quad (71)$$

The dynamical or flux expression for momentum loss is obtained by taking the prime derivative of Eq.(71) and eliminating the $\xi_R^{k''}$ by using Eq.(70), i.e.,

$$P^{k'} = \frac{1}{3}c^{-4}D_E^{i''}D_M^{j''}\epsilon_{ijk}. \quad (72)$$

which is just a different form of Eq.(70).

4.4 Physical Interpretation

We thus see that to second order in the perturbation, the imaginary part of the center of mass world-line leads to exactly the same equation for the radiated angular momentum as in the Schwarzschild case but, now, with a first-order (smaller) mass we do obtain recoil. The (real) center of mass (or the linear momentum) satisfies a 2^{nd} order evolution equation that is (similar to) Newton's 2^{nd} law with a recoil force and a radiation reaction force. In some sense we see that general relativity contains Newton's 2^{nd} law of motion.

In both perturbation schemes there is an anomalous contribution to the total angular momentum, namely in Eqs.(49) and (68) we see the term proportional to the rate of change of the magnetic dipole moment, i.e., D'_M . This can be considered as a prediction though how to measure it is not clear.

The two perturbations schemes thus lead to different physical consequences.

5 Perturbed Metric

For completeness, we display, to second order, the perturbed spin-coefficients and metric for the Schwarzschild background calculation. This metric, given in Bondi coordinates, is obtained by integrating the Bianchi identities, then integrating the spin coefficient and metric equations and finally constructing the metric[2]. To begin, we display the leading-order radial behavior for the

spin coefficients; the more complicated terms are bundled together and given in the Appendix.

$$\kappa = \varepsilon = \pi = 0 \quad (73)$$

$$\rho = -\frac{1}{r} - \frac{k\phi_0^0\bar{\phi}_0^0}{3r^5} \quad (74)$$

$$\sigma = \frac{\sigma^0}{r^2} \quad (75)$$

$$\tau = -\frac{\psi_1^0}{2r^3} + \mathfrak{S}_\tau \quad (76)$$

$$\alpha = \frac{\alpha^0}{r} - \frac{\beta^0\bar{\sigma}^0}{r^2} + \mathfrak{S}_\alpha \quad (77)$$

$$\beta = \frac{\beta^0}{r} - \frac{\alpha^0\sigma^0}{r^2} - \frac{\psi_1^0}{2r^3} + \mathfrak{S}_\beta \quad (78)$$

$$\gamma = -\frac{\psi_2^0}{2r^2} + \frac{\bar{\partial}\psi_1^0}{3r^3} + \mathfrak{S}_\gamma \quad (79)$$

$$\lambda = \frac{\lambda^0}{r} + \frac{\bar{\sigma}^0}{r^2} + \mathfrak{S}_\lambda \quad (80)$$

$$\mu = -\frac{1}{r} - \frac{\psi_2^0}{r^2} + \frac{\bar{\partial}\psi_1^0}{2r^3} + \mathfrak{S}_\mu \quad (81)$$

$$\nu = -\frac{\psi_3^0}{r} + \frac{\bar{\partial}\psi_2^0}{2r^2} - \frac{(\bar{\psi}_1^0 + \bar{\partial}^2\psi_1^0)}{6r^3} + \mathfrak{S}_\nu \quad (82)$$

Non-radial integrating factors in these expressions can be determined from the relations:

$$\begin{aligned} \alpha^0 &= -\bar{\beta}^0 = \frac{-\zeta}{2}, \\ \lambda^0 &= \sqrt{2}\bar{\sigma}^{0'}, \\ \sigma^0 &= \frac{\sqrt{2}k}{2c^3} \left[\frac{1}{3} \int (D^{i''}\bar{D}^{j''})du_r - (\bar{D}^i D^{j''})' - \frac{2}{3}(\bar{D}^{i''} D^{j''}) \right] Y_{2ij}^2, \\ \psi_3^0 &= \sqrt{2}\bar{\partial}\bar{\sigma}^{0'}, \\ \psi_4^0 &= -\sqrt{2}\lambda^{0'} = -2\bar{\sigma}^{0''} \end{aligned} \quad (83)$$

It is useful to define an auxiliary bundled expression, \mathfrak{A} , which enters into the metric expressions:

$$\begin{aligned} \mathfrak{A} = & \frac{\bar{\partial}\psi_1^0 + \bar{\partial}\bar{\psi}_1^0}{6r^2} - \frac{kq^2}{r^2} - kq \left(\frac{\sqrt{2}(D^{i'} + \bar{D}^{i'})}{r^2} + \frac{D^i - \bar{D}^i}{3r^3} \right) Y_{1i}^0 \\ & - k \left(\frac{10D^{i'}\bar{D}^{j'}}{9r^2} - \frac{(1+7\sqrt{2})(D^i\bar{D}^j)'}{27r^3} + \frac{13D^i\bar{D}^j}{45r^4} - \frac{8\sqrt{2}GM_S D^i\bar{D}^j}{45c^2r^5} \right) \delta_{ij} \\ & - ik \left(\frac{2D^i\bar{D}^{j'} - (D^i\bar{D}^j)'}{3r^3} \right) \epsilon_{ijk} Y_{1k}^0 - k \left(\frac{2D^{i'}\bar{D}^{j'}}{3r^2} + \frac{5\sqrt{2}D^i\bar{D}^{j'}}{18r^3} \right) Y_{2ij}^0 \\ & - k \left(\frac{(1-4\sqrt{2})(D^i\bar{D}^j)'}{108r^3} - \frac{7D^i\bar{D}^j}{90r^4} + \frac{2\sqrt{2}GM_S D^i\bar{D}^j}{45c^2r^5} \right) Y_{2ij}^0. \end{aligned} \quad (84)$$

The full covariant metric to second order in the Schwarzschild perturbation, takes the form

$$g_{ab} = \begin{bmatrix} g_{00} & 1 & g_{0\zeta} & g_{0\bar{\zeta}} \\ 1 & 0 & 0 & 0 \\ g_{0\zeta} & 0 & g_{\zeta\zeta} & g_{\zeta\bar{\zeta}} \\ g_{0\bar{\zeta}} & 0 & g_{\zeta\bar{\zeta}} & g_{\bar{\zeta}\bar{\zeta}} \end{bmatrix}$$

with:

$$\begin{aligned} g_{00} = & 2 + \frac{2}{r}(\Upsilon_\epsilon - \frac{2\sqrt{2}GM_S}{c^2}) + \frac{\sqrt{2}}{2r}(\psi_1^{0k'} + \bar{\psi}_1^{0k'})Y_{1k}^0 \\ & - \frac{2ik}{r}(D^{i'}\bar{D}^{j'})'\epsilon_{ijk}Y_{1k}^0 - 2\mathfrak{A} \\ & + \frac{k}{r}(\frac{\sqrt{2}}{2}(D^i\bar{D}^{j''} + \bar{D}^iD^{j''})' + \frac{\sqrt{2}}{3}(D^{i'}\bar{D}^{j'})')Y_{2ij}^0, \end{aligned} \quad (85)$$

$$\begin{aligned} g_{\zeta 0} = & P^{-1}[\bar{\partial}\sigma^0 + \frac{2\psi_1^{0k}Y_{1k}^1}{3r} - \frac{2kqD^iY_{1i}^1}{r^2}] \\ & - \frac{ik}{P}(\frac{2D^i\bar{D}^{j'}}{r^2} - \frac{4\sqrt{2}D^i\bar{D}^j}{15r^3})\epsilon_{ijk}Y_{1k}^1 \\ & + \frac{k}{P}(\frac{2D^i\bar{D}^{j''}}{r} - \frac{4\sqrt{2}D^i\bar{D}^{j'}}{3r^2} + \frac{4D^i\bar{D}^j}{15r^3})Y_{2ij}^1, \end{aligned} \quad (86)$$

$$\begin{aligned} g_{\bar{\zeta} 0} = & P^{-1}[\bar{\partial}\bar{\sigma}^0 + \frac{2\bar{\psi}_1^{0k}Y_{1k}^{-1}}{3r} - \frac{2kq\bar{D}^iY_{1i}^{-1}}{r^2}] \\ & + \frac{ik}{P}(\frac{2\bar{D}^iD^{j'}}{r^2} - \frac{4\sqrt{2}\bar{D}^iD^j}{15r^3})\epsilon_{ijk}Y_{1k}^{-1} \\ & + \frac{k}{P}(\frac{2\bar{D}^iD^{j''}}{r} - \frac{4\sqrt{2}\bar{D}^iD^{j'}}{3r^2} + \frac{4\bar{D}^iD^j}{15r^3})Y_{2ij}^{-1}, \end{aligned} \quad (87)$$

$$g_{\zeta\zeta} = -\frac{2\sigma^0 r}{P^2}, \quad g_{\bar{\zeta}\bar{\zeta}} = -\frac{2\bar{\sigma}^0 r}{P^2}, \quad (88)$$

$$g_{\zeta\bar{\zeta}} = \frac{-r^2}{P^2}. \quad (89)$$

and

$$P \equiv (1 + \zeta\bar{\zeta}).$$

The corresponding metric for the perturbation off of the Minkowski background is easily obtained from these results.

6 Discussion

We begin with a mea culpa. In this work we started with two given vacuum metrics, the Schwarzschild and Minkowski metrics and then ‘drove’ or perturbed them both with a ‘small’ Maxwell field. In one case there was a ‘large’ mass while in the second case the mass was ‘small’. There was no systematic attempt, by comparisons, to define small or large quantities; there was no small parameter for a series expansion. By ‘small’ we meant it to be as small as needed for a physical effect; or as large as was needed for a different effect. The idea behind the two perturbation schemes was purely heuristic. We wanted to see, in a rough sense, within the *context of the Einstein equations*[10], what would be the physical responses of the recently defined (complex) *center of mass* of massive systems (with different size masses) to an electromagnetic perturbation. In addition to the totally expected result that the large mass object did not experience any recoil while the small object did experience recoil, we obtained a variety of concomitant physical results. The imaginary part of the (complex) *center of mass world-line* could be identified with the electrodynamic induced internal spin angular momentum that satisfies the classical (angular momentum) conservation law. There was an electromagnetically induced gravitational radiation as well as linear momentum loss. We obtained explicit equations of motion for the center of mass that have the form of Newton’s 2nd law.

The results presented here, in the two different models, are in total agreement with well-known classical Newtonian and electrodynamics effects. They however go beyond these known results. They give a confirmation of the physical ideas developed purely in the context gravitational theory (general relativity) for the identification and the associated dynamical response of certain geometric structures that arise naturally. The most important of them is the special class of null geodesic congruences, the shear-free (or asymptotically shear-free) null geodesic congruences. It is their very existence that yields or provides the complex Minkowski world-lines that become our complex center-of mass which then yield our angular momentum expressions.

As a final comment we point out that the perturbed space-times described here contain, as special cases, both the Kerr and the charged Kerr metrics as

perturbations off Schwarzschild. They occur when the electromagnetic dipoles are ‘shut off’, i.e., when $D^i = 0$.

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8 References

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9 Appendix

9.1 Radial Behavior of the Weyl Tensor

Below are the full expressions for the bundle terms (\mathcal{A}_i) given in Eq. 5-8:

$$\begin{aligned} \mathcal{A}_1 = & -\frac{5k}{r^5}[2qD^iY_{1i}^1 + 2\sqrt{2}D^i\bar{D}^{j'}(\frac{i}{\sqrt{2}}\epsilon_{ijk}Y_{1k}^1 + \frac{1}{2}Y_{2ij}^1)] \\ & + \frac{6kD^i\bar{D}^j}{r^6}(\frac{i}{\sqrt{2}}\epsilon_{ijk}Y_{1k}^1 + \frac{1}{2}Y_{2ij}^1) \end{aligned} \quad (90)$$

$$\begin{aligned} \mathcal{A}_2 = & -\frac{\bar{\partial}\psi_1^0}{r^4} + \frac{2kq^2}{r^4} + kq(\frac{2\sqrt{2}(D^{i'} + \bar{D}^{i'})}{r^4} + \frac{10D^i - 2\bar{D}^i}{3r^5})Y_{1i}^0 \\ & + 2k(\frac{D^{i'}\bar{D}^{j'}}{r^4} - \frac{(1 + 4\sqrt{2})(D^i\bar{D}^j)'}{9r^5} + \frac{10D^i\bar{D}^j}{9r^6} - \frac{8\sqrt{2}GM_S D^i\bar{D}^j}{3c^2r^7})\delta_{ij} \\ & + ik(\frac{12D^i\bar{D}^{j'}}{3r^5} + \frac{(4 + \sqrt{2})(D^i\bar{D}^j)'}{6r^5} - \frac{\sqrt{2}D^i\bar{D}^j}{r^6})\epsilon_{ijk}Y_{1k}^0 \\ & + k(\frac{4D^{i'}\bar{D}^{j'}}{3r^4} + \frac{30\sqrt{2}D^i\bar{D}^{j'} + (1 + 2\sqrt{2})(D^i\bar{D}^j)'}{18r^5})Y_{2ij}^0 \\ & - k(\frac{10D^i\bar{D}^j}{9r^6} - \frac{2\sqrt{2}GM_S D^i\bar{D}^j}{3c^2r^7})Y_{2ij}^0 \end{aligned} \quad (91)$$

$$\begin{aligned}
\mathcal{A}_3 = & -\frac{\bar{\partial}\psi_2^0}{r^3} + \frac{\bar{\partial}^2\psi_1^0}{2r^4} - \frac{2kqD^{i''}}{r^3}Y_{1i}^{-1} \\
& + 2kq\left(\frac{(1+7\sqrt{2})D^{i'}+6\sqrt{2}\bar{D}^{i'}}{3r^4} + \frac{(11D^i+12\bar{D}^i)}{18r^5} - \frac{2\sqrt{2}GM_S\bar{D}^i}{c^2r^6}\right)Y_{1i}^{-1} \\
& + ik\left(\frac{2D^{i''}\bar{D}^{j'}}{r^3} - \frac{2\sqrt{2}D^{i'}\bar{D}^{j'}+(2-2\sqrt{2})(\bar{D}^iD^{j'})'}{3r^4}\right)\epsilon_{ijk}Y_{1k}^{-1} \\
& + ik\left(\frac{(34-\sqrt{2})(D^i\bar{D}^j)'-D^i\bar{D}^{j'}}{9r^5} + \frac{5\sqrt{2}\bar{D}^iD^j}{4r^6} - \frac{56GM_S\bar{D}^iD^j}{5c^2r^7}\right)\epsilon_{ijk}Y_{1k}^{-1} \\
& - k\left(\frac{\sqrt{2}D^{i''}\bar{D}^{j'}}{r^3} - \frac{20D^{i'}\bar{D}^{j'}+\sqrt{2}(\bar{D}^iD^{j'})'}{3r^4}\right)Y_{2ij}^{-1} \\
& - k\left(\frac{37\sqrt{2}D^{i'}\bar{D}^j+(3-49\sqrt{2})(D^i\bar{D}^j)'}{18r^5} + \frac{23D^i\bar{D}^j}{12r^6}\right)Y_{2ij}^{-1} \\
& + \frac{2\sqrt{2}Gk}{c^2}\left(\frac{\sqrt{2}M_S\bar{D}^iD^{j'}}{r^6} + \frac{16M_SD^i\bar{D}^j}{5r^7}\right)Y_{2ij}^{-1}
\end{aligned} \tag{92}$$

$$\begin{aligned}
\mathcal{A}_4 = & -\frac{\bar{\partial}\psi_3^0}{r^2} + \frac{\bar{\partial}^2\psi_2^0}{2r^3} - \frac{\bar{\partial}^3\psi_1^0}{6r^4} \\
& - k\left(\frac{3\sqrt{2}D^{i''}\bar{D}^{j'}+2(D^{i''}\bar{D}^j)'}{r^3} - \frac{(16\sqrt{2}-72)(D^{i'}\bar{D}^j)'}{9r^4}\right) \\
& - \frac{128D^{i'}\bar{D}^{j'}}{9r^4} + \frac{(24-107\sqrt{2})(D^i\bar{D}^j)'-25\sqrt{2}D^{i'}\bar{D}^j}{36r^5} + \frac{47D^i\bar{D}^j}{15r^6} \\
& + \frac{240GM_SD^{i'}\bar{D}^j}{15c^2r^6} - \frac{38\sqrt{2}GM_SD^i\bar{D}^j}{5c^2r^7}Y_{2ij}^{-2}.
\end{aligned} \tag{93}$$

9.2 The Spin Coefficients

The following are the full expressions for the \mathfrak{S}_i given in Section V for the spin coefficients in the Schwarzschild background perturbation:

$$\begin{aligned}
\mathfrak{S}_\tau = & \frac{8kqD^i}{3r^4}Y_{1i}^1 + ik\left(\frac{8D^i\bar{D}^{j'}}{3r^4} - \frac{\sqrt{2}D^i\bar{D}^j}{2r^5}\right)\epsilon_{ijk}Y_{1k}^1 \\
& + k\left(\frac{4\sqrt{2}D^i\bar{D}^{j'}}{3r^4} - \frac{D^i\bar{D}^j}{2r^5}\right)Y_{2ij}^1
\end{aligned} \tag{94}$$

$$\begin{aligned}
\mathfrak{S}_\alpha = & -\frac{2kq\bar{D}^i}{3r^4}Y_{1i}^{-1} + ik\left(\frac{2\bar{D}^iD^{j'}}{3r^4} - \frac{\sqrt{2}\bar{D}^iD^j}{4r^5}\right)\epsilon_{ijk}Y_{1k}^{-1} \\
& - k\left(\frac{\sqrt{2}\bar{D}^iD^{j'}}{3r^4} - \frac{\bar{D}^iD^j}{4r^5}\right)Y_{2ij}^{-1} + \frac{k\alpha^0\phi_0^0\bar{\phi}_0^0}{12r^5}
\end{aligned} \tag{95}$$

$$\begin{aligned}\mathfrak{S}_\beta &= \frac{10kqD^i}{3r^4}Y_{1i}^1 + ik\left(\frac{10D^i\bar{D}^{j'}}{3r^4} - \frac{\sqrt{2}D^i\bar{D}^j}{2r^5}\right)\epsilon_{ijk}Y_{1k}^1 \\ &\quad + k\left(\frac{5\sqrt{2}D^i\bar{D}^{j'}}{3r^4} - \frac{D^i\bar{D}^j}{2r^5}\right)Y_{2ij}^1 + \frac{k\beta^0\phi_0^0\bar{\phi}_0^0}{12r^5}\end{aligned}\quad (96)$$

$$\begin{aligned}\mathfrak{S}_\gamma &= \frac{\alpha^0\psi_1^0 + \beta^0\bar{\psi}_1^0}{6r^3} - \frac{kq^2}{r^3} - kq\left(\frac{\sqrt{2}(D^{i'} + \bar{D}^{i'})}{r^3} + \frac{7D^i - 5\bar{D}^i}{12r^4}\right)Y_{1i}^0 \\ &\quad - k\left(\frac{10D^{i'}\bar{D}^{j'}}{9r^3} - \frac{(1+7\sqrt{2})(D^i\bar{D}^j)'}{18r^4} + \frac{26D^i\bar{D}^j}{45r^5} + \frac{2MD^i\bar{D}^j}{9r^6}\right)\delta_{ij} \\ &\quad - ik\left(\frac{24D^i\bar{D}^{j'} + (4+\sqrt{2})(D^i\bar{D}^j)'}{24r^4} - \frac{\sqrt{2}D^i\bar{D}^j}{6r^6}\right)\epsilon_{ijk}Y_{1k}^0 \\ &\quad - k\left(\frac{2D^{i'}\bar{D}^{j'}}{3r^3} + \frac{(1-4\sqrt{2})(D^i\bar{D}^j)' + 30\sqrt{2}D^i\bar{D}^{j'}}{24r^4} - \frac{7D^i\bar{D}^j}{45r^5} - \frac{MD^i\bar{D}^j}{18r^6}\right)Y_{2ij}^0 + \mathfrak{B}\end{aligned}\quad (97)$$

$$\begin{aligned}\mathfrak{B} &\equiv -\frac{2kq}{3r^4}(\alpha^0D^iY_{1i}^1 + \beta^0\bar{D}^iY_{1i}^{-1}) - ik\alpha^0\left(\frac{8D^i\bar{D}^{j'}}{9r^3} - \frac{\sqrt{2}D^i\bar{D}^j}{16r^4}\right)\epsilon_{ijk}Y_{1k}^1 \\ &\quad - k\alpha^0\left(\frac{4\sqrt{2}D^i\bar{D}^{j'}}{9r^3} - \frac{D^i\bar{D}^j}{16r^4}\right)Y_{2ij}^1 + ik\beta^0\left(\frac{8\bar{D}^iD^{j'}}{9r^3} - \frac{\sqrt{2}\bar{D}^iD^j}{16r^4}\right)\epsilon_{ijk}Y_{1k}^{-1} \\ &\quad - k\beta^0\left(\frac{4\sqrt{2}\bar{D}^iD^{j'}}{9r^3} - \frac{\bar{D}^iD^j}{16r^5}\right)Y_{2ij}^{-1}\end{aligned}\quad (98)$$

$$\mathfrak{S}_\lambda = k\left(\frac{2D^{i''}\bar{D}^j}{r^3} - \frac{4\sqrt{2}D^{i''}\bar{D}^j}{3r^4} + \frac{D^i\bar{D}^j}{2r^5}\right)Y_{2ij}^{-2}\quad (99)$$

$$\begin{aligned}\mathfrak{S}_\mu &= -\frac{kq^2}{r^3} - kq\left(\frac{\sqrt{2}(D^{i'} + \bar{D}^{i'})}{r^3} + \frac{10D^i - 2\bar{D}^i}{9r^4}\right)Y_{1i}^0 \\ &\quad - k\left(\frac{D^{i''}\bar{D}^{j'}}{r^3} - \frac{(2+8\sqrt{2})(D^i\bar{D}^j)'}{27r^4} + \frac{5D^i\bar{D}^j}{3r^5} - \frac{8\sqrt{2}GM_S D^i\bar{D}^j}{15c^2r^6}\right)\delta_{ij} \\ &\quad - ik\left(\frac{4D^i\bar{D}^{j'}}{3r^4} + \frac{(4+\sqrt{2})(D^i\bar{D}^j)'}{18r^4} - \frac{\sqrt{2}D^i\bar{D}^j}{4r^5}\right)\epsilon_{ijk}Y_{1k}^0 - \frac{2\sqrt{2}kGM_S D^i\bar{D}^j}{15c^2r^6}Y_{2ij}^0 \\ &\quad - k\left(\frac{2D^{i''}\bar{D}^{j'}}{3r^3} + \frac{(1+2\sqrt{2})(D^i\bar{D}^j)' + 30\sqrt{2}D^i\bar{D}^{j'}}{54r^4} - \frac{10D^i\bar{D}^j}{36r^5}\right)Y_{2ij}^0\end{aligned}\quad (100)$$

$$\begin{aligned}
\mathfrak{S}_\nu = & -ik\left(\frac{2D^{i''}\bar{D}^{j'}}{r^2} - \frac{8\sqrt{2}D^{i'}\bar{D}^{j'} + \sqrt{2}D^{i''}\bar{D}^j + (1-\sqrt{2})(\bar{D}^i D^{j'})'}{9r^3}\right)\epsilon_{ijk}Y_{1k}^{-1} \\
& -ik\left(\frac{8D^i\bar{D}^{j'} + 6\bar{D}^i D^{j'} + (34-\sqrt{2})(D^i\bar{D}^j)'}{36r^4} - \frac{7\sqrt{2}\bar{D}^i D^j}{20r^5}\right)\epsilon_{ijk}Y_{1k}^{-1} \\
& -ik\left(\frac{8M\bar{D}^i D^{j'}}{15r^5} - \frac{92GM_S\bar{D}^i D^j}{60c^2r^6}\right)\epsilon_{ijk}Y_{1k}^{-1} + k\left(\frac{\sqrt{2}D^{i''}\bar{D}^{j'}}{r^2}\right)Y_{2ij}^{-1} \\
& -k\left(\frac{3D^{i''}\bar{D}^j + 26D^{i'}\bar{D}^{j'} + \sqrt{2}(\bar{D}^i D^{j'})'}{9r^3} - \frac{70\sqrt{2}D^{i'}\bar{D}^j + (3-40\sqrt{2})(D^i\bar{D}^j)'}{72r^4}\right)Y_{2ij}^{-1} \\
& +k\left(\frac{11c^2\bar{D}^i D^j - 16GM_S\bar{D}^i D^{j'}}{60c^2r^5} - \frac{54\sqrt{2}GM_S D^i\bar{D}^j}{60c^2r^6}\right)Y_{2ij}^{-1}.
\end{aligned} \tag{101}$$